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Single Peakedness and Giffen Demand

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Abstract

I provide a simple example of a single peaked utility function that generates a Giffen demand. The utility function is smooth, non piecewise defined, strictly concave but not globally increasing. A full characterization of the parameter conditions under which the Giffen demand arises is provided. In addition the properties of the demand function are studied: I find that the inferior commodity with a Giffen demand must be cheaper relatively to a substitute and that Giffen demand arises at relatively low levels of income. However it is not required that the share of income spent on that commodity be large.

Keywords: Giffen behavior; Utility function; Single peakedness; Expenditure share.

JEL code: D11

1 Introduction

The existence of Giffen goods beyond the theoretical possibility that was then formalized in the Slutsky equation, has been questioned for a long time (see, for example, Stigler, 1947; Vandermeulen, 1972). Recently, Jensen and Miller (2008), have found evidence of Giffen behavior for consumption of rice in rural China. [1] This discovery brings back to relevance the understanding of the micro foundations of such a demand function. [2]

In addition, Giffen goods are part of most microeconomics syllabi and the question about which utility function can generate them is always asked. It is therefore an interesting exercise to analyze which preferences and utility functions can generate demands with the Giffen property.

This note provides an example of a simple, smooth and conventional utility function that generates a Giffen demand, and explores the properties of such a demand in comparison with what "conventional wisdom" lead us believe. In particular, it is found that the Giffen demand arises for relatively cheap commodities who have a more expensive substitute; however, a Giffen demand may arise even at low levels of expenditure, relative to income, for that commodity.

The utility function presented here is simple since it is given by a quadratic single peaked function, which is therefore smooth. It is conventional because single peaked utility functions are now more adopted in the Economic literature, from Political Economy (see, e.g., Persson and Tabellini, 2000) to Empirical Industrial Organization (see, e.g., Goettler and Shachar, 2001).

Unlike standard textbook utilities, single peaked ones are not globally monotone, and so they clash with the standard notion that individual preferences are characterized by unlimited wants. However single peaked utilities represent strictly convex preferences, another standard notion in consumer's theory which is not always present in the ex-

[1] In particular, they provide evidence that demand for rice has the Giffen property amongst poor people who spend a quite significantly large share of their income in it and for which there are some substitutes who are more expensive.

[2] The recent works by Doi et al. (2009) and Heijman and Mouche (2012) provide an exhaustive and well crafted introduction to this topic.

amples of utilities generating Giffen demand provided so far. Last, as we will see, all the action takes place when the bliss point is unaffordable, and therefore in the subset of the commodity space where standard tangency analysis applies.

It has been already found (Spiegel, 1994; Vandermeulen, 1972; Weber, 1997) that satiation for at least one commodity is a key ingredient to generate demand curves that are increasing in price. However the examples of utility functions provided might not be too compelling: Vandermeulen (1972) does not generate a continuous demand function, Spiegel (1994) and Heijman and Mouche (2012) rely on non convex preferences, while Weber (1997) uses a quite unconventional utility function. [3]

Interestingly, Giffeneity of the demand function can also arise under concave and locally satiated preferences (Butler and Moffatt, 2000). In this case, both monotonicity and convexity of preferences are violated, so that completely non standard utility functions are analyzed. With concave preferences corner solutions are expected, but by introducing some complementarity between the two commodities Butler and Moffatt (2000) can obtain interior solutions to the utility maximization problem and, in addition, they find that the inferior commodity becomes a Giffen good when it is cheaper than its superior substitute and income is at intermediate values.

The possibility that monotone and convex preferences generate demands that are upwards sloping are discussed by Moffatt (2002), Sørensen (2007), and Doi et al. (2009). Moffatt (2002) constructs a family of indifference curves (hyperboles) starting from the price offer curve, that are increasing and convex, and have the Giffen property. While Moffatt does not give any functional form for the underlying utility function, Heijman and Mouche (2012) and Moffatt (2012) use his approach to provide some examples, which are not that simple nor conventional.

Sørensen (2007), instead, modifies Leontieff utility functions by introducing some degree of substitutability between the two commodities. Unlike Sørensen, I obtain smooth indifference curves, which can be easily plotted. This allows to obtain the Giffen demand via simple graphical analysis, through the price consumption path, and standard

[3] The indifference curves adopted in Jensen and Miller (2008) to show a theoretical possibility for Giffen behavior resembles those used by Spiegel (1994).

calculus techniques, through the first order conditions.

Doi et al. (2009) provide an example of a piecewise monotone and convex utility function that generates a Giffen demand, and offer a complete a characterization of the conditions under which a Giffen demand arises. They then find that the Giffen demand generated by their example is compatible with an arbitrarily low share of income spent on that commodity. Unlike them, I forgo global monotonicity of preferences but I obtain a Giffen demand from a very simple, non piecewise, smooth utility function which is commonly used in the literature.

This is valuable in abstract modeling of consumers' behavior (since parsimony should rule), in empirical tests of consumers' demands, as single peaked utility functions have been already used (Goettler and Shachar, 2001), and in the classroom, where this example of utility generating Giffen demand can be presented at any level. In addition I find that the Giffen demand might or might not involve a commodity that exhausts a large share of the consumers' budget.

To the best of my knowledge, this is the first analysis of convex but not monotone preferences. The result is striking because of the simplicity of the utility function that is needed to generate a Giffen demand.

The following section provides a numerical example, while the main analysis of the paper is in Section 3; Section 4 provides some other numerical applications, while Section 5 concludes. All the results follow from simple algebraic manipulation and, therefore, are relegated to the Appendix.

2 A numerical example

Consider the following utility function over (units of) commodities x and y, subsets of $\mathbb{R}^2_+$:

$$u(x, y) = -(x^2 + y^2) + 19.6(x + y) - 0.8xy - 137.2$$

This function is strictly concave in its arguments and has a global maximum at $x = y = 7$. For any other possible pair (x, y) in $\mathbb{R}^2_+$, we have that $u(x, y) < u(7, 7) = 0$. In

addition, the indifference curves are ellipsis around the bliss point $(7,7)$.
Let the unit price of x be denoted by p. Assume the unit price of y is 1 and the income available for purchase is 6. Note that at the current income the bliss point is not affordable no matter how low p is, i.e. even at $p=0$. Because of that, it is possible to derive the demand curve of x via simple graphical analysis, as shown in Figure 1.

[Figure 1 about here.]

The top panel shows the elliptical indifference curves, different budget lines as p varies and the price consumption path. The bottom panel indicates the demand for x so derived. We can see that demand is upward sloping and convex for a non empty and convex subset of the prices.
In the next section I provide a complete characterization of the conditions under which these features emerge from a family of quadratic and single peaked utility functions.

3 The family of utility functions

Consider the following utility function defined in $\mathbb{R}^2_+$

$$u(x,y) = -a(x^2+y^2) + b(x+y) - cxy - d \tag{1}$$

where $a>0$, $b>0$, $c>0$ and $d>0$. In what follows I assume that $2a>c$ so as to guarantee strict concavity. In this case the utility has a unique global maximum given by $(\frac{b}{2a+c}, \frac{b}{2a+c})$. To minimize on notation, from now onwards I normalize $a=1$. In addition we can set d equal to zero or such that the maximum utility is equal to zero (hence $d=\frac{b^2}{2+c}$).
When $c>0$ the utility function (1) has elliptical indifference curves, whereas when $c=0$ it has circular indifference curves. Most of the economic applications with single peaked utility functions involve $c=0$.
Let the unit price of x and y be given by, respectively, p and q. As the analysis is

done entirely on the demand for x, I can adopt the normalization $q = 1$. [4] Last, let M denote the income. The budget constraint becomes $px + y \leq M$.

If the bliss point is affordable, i.e. $(p+1)\frac{b}{2+c} \leq M$, then it is the solution of the constrained utility maximization problem. Otherwise, at the optimum the budget constraint must be binding and therefore a standard tangency analysis applies, save for the case of corner solutions. As a result the demand for x is given by

$$x^d(p,M) = \begin{cases} \frac{b}{2+c} & \text{if } (p+1)b \leq M(2+c) \\ \min\left\{\frac{M}{p}, \max\left\{0, \frac{(b-Mc)-(b-2M)p}{2(1-cp+p^2)}\right\}\right\} & \text{Otherwise} \end{cases} \tag{2}$$

In an analogous fashion one can find the demand for y, which is given by

$$y^d(p,M) = \begin{cases} \frac{b}{2+c} & \text{if } (p+1)b \leq M(2+c) \\ \min\left\{M, \max\left\{0, \frac{(bp-Mc)p-(bp-2M)}{2(1-cp+p^2)}\right\}\right\} & \text{Otherwise} \end{cases} \tag{3}$$

Unlike the standard case where preferences exhibit local non-satiation, we need to take into account the possibility of corner solutions (spend nothing or all the income in either one of the commodities). This explains the need for the min and max conditions in the demand functions.

In particular, any time $y^d(p,M) > 0$ we have that $x^d(p,M) < M/p$, and any time $x^d(p,M) > 0$ we have that $y^d(p,M) < M$. Finally note that $1 - cp + p^2 > 0$ as $c < 2$. Therefore both demands are interior any time $(b - Mc) - (b - 2M)p > 0$ and $(bp - Mc)p - (bp - 2M) > 0$.

Note that the possibility of corner solutions complicates the analysis quite a bit. However, once the parameters values generating a Giffen demand are found, the equation of the utility function and the subsequent calculations are simple.

The following proposition characterizes the conditions under which we are in the presence of a normal or an inferior good:

Proposition 1. *Suppose $(p+1)b > M(2+c)$ and both $x^d(p,M)$ and $y^d(p,M)$ are*

[4] Since the utility chosen is symmetric, if the whole analysis is carried over commodity y little is added at the expenses of keeping track of q.

interior. Commodity y is normal and commodity x is inferior if and only if $p < c/2$; both commodities are normal if and only if $p \in [c/2, 2/c]$; commodity y is inferior and commodity x is normal if and only if $p > 2/c$.

Proposition 1 follows from the fact that

$$\frac{\partial x^d(p,M)}{\partial M} = \frac{2p-c}{2(p^2+1-cp)} \tag{4}$$

and

$$\frac{\partial y^d(p,M)}{\partial M} = \frac{2-cp}{2(p^2+1-cp)} \tag{5}$$

and the denominator is always positive whenever $c < 2$. This result differs from Doi et al. (2009), according to which demand for y is always normal, and it is due to our specification of single peaked utility function. Proposition 1 states that we can have at most one inferior good when the budget constraint is binding, which is a familiar result. Note also that a necessary condition for x to be inferior is that $p < 1$.

I now focus on the demand for x. It is a straightforward application of the Maximum theorem to show that x^d is continuous in p and M. In fact, this demand is well defined and continuous even at $p = 0$, since the consumer does not want more of $b/(2+c)$ of x. Last, this demand is differentiable virtually everywhere. The following proposition states the main result of this paper.

Proposition 2. *Let*

$$p^\star = M\frac{2+c}{b} - 1, \tag{6}$$

$$\tilde{p} = \frac{b-cM}{b-2M} - \sqrt{\left(\frac{b-cM}{b-2M}\right)^2 - \frac{c(b-cM)}{b-2M} + 1} \tag{7}$$

and

$$\bar{p} = \frac{b+cM-\sqrt{(b+cM)^2-8bM}}{2b} \tag{8}$$

The demand for x has the Giffen property if and only if one of the following holds

1. $p \in (0, \min\{\tilde{p}, \bar{p}\})$, $\sqrt{3} \le c < 2$, *and* $M < \frac{b}{2+c}$;

2. $p \in (0, \tilde{p})$, $1 \le c < \sqrt{3}$, *and* $\frac{b(4-c)}{c^2} - \frac{2b}{c^2}\sqrt{2(2-c)} < M < \frac{b}{2+c}$*;*

3. $p \in (0, \min\{\tilde{p}, \bar{p}\})$, $1 \le c < \sqrt{3}$, *and* $0 < M \le \frac{b(4-c)}{c^2} - \frac{2b}{c^2}\sqrt{2(2-c)}$*;*

4. $p \in (0, \tilde{p})$, $\frac{\sqrt{17}-1}{4} < c < 1$ *and* $\frac{b(4-c)}{c^2} - \frac{2b}{c^2}\sqrt{2(2-c)} < M < \frac{b}{2+c}$*;*

5. $p \in (0, \min\{\tilde{p}, \bar{p}\})$, $\frac{\sqrt{17}-1}{4} < c < 1$ *and* $\frac{b(1-c)}{2-c^2} < M \le \frac{b(4-c)}{c^2} - \frac{2b}{c^2}\sqrt{2(2-c)}$*;*

6. $p \in (0, \tilde{p})$, $0 < c \le \frac{\sqrt{17}-1}{4}$ *and* $\frac{b(1-c)}{2-c^2} < M < \frac{b}{2+c}$*;*

7. $p \in (p^\star, \tilde{p})$, $\sqrt{3} < c < 2$, *and* $\frac{b(4-c)}{c^2} - \frac{2b}{c^2}\sqrt{2(2-c)} < M < \frac{b}{2}$.

8. $p \in (p^\star, \min\{\tilde{p}, \bar{p}\})$, $\sqrt{3} < c < 2$, *and* $\frac{b}{c+2} < M \le \frac{b(4-c)}{c^2} - \frac{2b}{c^2}\sqrt{2(2-c)}$.

9. $p \in (p^\star, \tilde{p})$, $0 < c \le \sqrt{3}$, *and* $\frac{b}{2+c} < M < \frac{b}{2}$.

In addition, whenever x is a Giffen good, $\tilde{p} < 1$.

Last, the demand of y is decreasing in p whenever x is a Giffen good, but has an uncertain sign in general.

Few comments about the cutoff values for p so obtained. In the statement of the Proposition we need to look at the parameter values for which the demand for x is interior when the bliss point is not affordable: $p^\star$ tells the minimum price at which the bliss point becomes unaffordable. When, given the values of b, c, and M, $p^\star < 0$, then we know that the bliss point is never affordable. When instead $p^\star > 0$, prices cannot go below it if we want to find a Giffen demand; $\bar{p}$ sets the maximum value of prices for which the demand for x is below M/p, the maximum amount that can be purchased of it.

Last note that $\tilde{p} < 1$ follows from $c < 2$ and $M < b/2$. In fact, under these assumptions, $\frac{b-cM}{b-2M} > 1$ and in addition $\tilde{p} < 1$.

3.1 Properties of the Giffen demand

Proposition 2 has two main conditions. The one involving the vertical intercept of the budget line, that poses an upper bound on the income of the consumer; and the one involving the price level p, which poses an upper bound (and sometimes a lower bound) on the unit price of x.

The condition on the income requires M be not too large. Specifically, we have three upper bounds on M: $b(4-c)/c^2 - 2b\sqrt{4-2c}/c^2$, $b/(2+c)$, and $b/2$. The first is needed to ensure that $x^d(p, M) < M/p$. The condition $M < b/2$ means that the vertical intercept of the budget line is below the preferred quantity of y computed when $x = 0$. Therefore it imposes that the bundle $(0, \frac{b}{2})$ be unaffordable. The condition, $M < b/(2+c)$ means that the vertical intercept of the budget line is below the y component of the bliss point. Therefore the bliss point becomes unaffordable for all values of p. Hence, conditions 1 to 6 require that the bliss point be unaffordable for all p, while conditions 7 to 9 require the bliss point be affordable in a neighborhood of $p = 0$, while the optimal value of y at $x = 0$ is never affordable.

The condition on the unit price of x requires $p < 1$. From the assumption $c < 2$ and the condition $M < b/2$ follow that $\tilde{p}$, the upper bound of the values of p for which a Giffen demand is obtained, is smaller than q. This result is expected, since Proposition 1 shows that $p < 1 (= q)$ is a necessary condition for x to be inferior. [5]

Overall these conditions match quite well with what Economists thought a good candidate for a Giffen good would be (see, e.g. Mas-Colell et al., 1995, page 26), which is also what Jensen and Miller (2008) have recently discovered empirically. Namely, the commodity must be relatively cheap compared to some superior substitute and the total income must be relatively small.

3.2 Share of income spent on the Giffen good

The next result shows that there is a main difference with such a common wisdom: the Giffen good does not require a large expenditure of the consumer's income. This result corroborates the critique done by Vandermeulen (1972) and has also been found by Doi et al. (2009), who showed that the Giffen demand provided by their example is compatible with an arbitrarily small share of income spent on that commodity.

[5] These results do not rely on the normalization $q = 1$ chosen, since they can all be expressed in terms of price ratio $p/q < 1$ in a more general setting.

However, in this case we have that a Giffen good is consistent with a more complex composition of consumer's expenditures, that allows for small or large share of income purchased on the inferior commodity.

Proposition 3. *Suppose x is a Giffen good. For the parameter values corresponding to cases 1 to 6 in Proposition 2 the share of income spent in x can be made arbitrarily small. In addition, the share of income spent in x is always less than $1/2$ in cases 4 to 7 of Proposition 2 and it can be smaller than $1/2$ in the remaining ones if c is not too big (cases 2 and 9), or if both c and M are sufficiently large (cases 2, 4 and 9), or if c is sufficiently large and p is sufficiently small (cases 1 and 8).*

The first result in Proposition 3 is due to the fact that with a non globally monotone utility, it is possible to have a maximum quantity demanded that is finite, and therefore the expenditure in one commodity can be made arbitrarily small by decreasing its unit price. Since a condition for Giffen demand is that p be small, and possibly zero, the result follows.

Note also that Proposition 3 also implies that there might be parameter values for which x is a Giffen good and its expenditure is more than half of the budget, which happens when prices and income are not too small. This provides only partly a justification for the common wisdom on Giffen goods that they exhaust a large share of the consumer's budget. (see, e.g. Mas-Colell et al., 1995, page 26)

As we have seen from the numerical example in Section 2, $x^d(p, M)$ can be concave whenever x is a Giffen good (see also Figure 1). This is quite a general result as the following proposition states.

Proposition 4. *There exists a non empty subset of the level of M and c for which x is a Giffen good and its demand is increasing and concave for a convex and non empty subset of the prices.*

Proposition 4 addresses a remark of Heijman and Mouche (2012), who have noticed the rarity of a concave demand function in the price range for which the commodity is a Giffen good.

4 Some other graphical examples

Proposition 2 allows for a Giffen demand to emerge under several parameter configurations, some of which do not generate a straightforward analysis (even graphically). Figures 2 and 3 report some possible possible scenarios.

[Figure 2 about here.]

[Figure 3 about here.]

As we can see, the derivation of the demand curve can become more complex. Again this is due to the existence of corner solutions (Figure 2) or to fact that the bliss point is affordable (Figure 3) for some prices, which can occur when preference are not monotone. Nevertheless things can be kept also quite simple and a standard analysis can be carried over, as shown in Figure1.

5 Conclusion

This note provides an example of utility functions that generate a demand with the Giffen property. This demand can be obtained via standard calculus or graphical analysis. The class of utility functions analyzed is quite common in the Economics literature, and easy to handle analytically.

The derived demand curve shows the Giffen effect at relatively low prices and income. Economists have often speculated about the possibility of a Giffen demand in that range of price and income. The common examples of Giffen goods, public versus private transportation, and cheap, less tasty versus expensive and tasty food (Spiegel, 1994), seem to suggest the existence of a satiation point and substitutability between the two commodities. The utility function presented here meet both requirements.

Recent development in the empirical literature (Jensen and Miller, 2008) also finds Giffen goods in a context that can be rationalized by the type of preferences presented in this note: the agent has low income and some substitutability between the two food items is needed.

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A Proof of the Propositions

Proof of Proposition 2. The proof consists of exploring a set of inequalities which are needed to impose that the demand of x be defined when the bliss point is not affordable, positive, bounded from above by M/p, and increasing.

The first steps involve analyzing the behavior of the demand of x without taking into account the requirement that $x^d(p, M) \leq M/p$. The final step involves incorporating this constraint into the set of conditions that have been found.

In order for $x^d(p, M)$ to show the Giffen property it must be that $x^d(p, M)$ is increasing in price. This requires $x^d(p, M)$ to be non negative and non constant. In particular the bliss point must be unaffordable.

Condition for bliss point not affordable: From equation (2) we see that the bliss point is unaffordable if and only if

$$p > \frac{M(2+c)}{b} - 1 \tag{9}$$

which is always true whenever the right hand side is non positive. Remark that this happens if and only if $b \geq M(2+c)$.

Condition for demand of x positive: From (2) we also see that the demand of x is positive if and only if, in addition, [6]

$$(b - Mc) - (b - 2M)p > 0 \tag{10}$$

Condition for demand of x below M/p.

Demand of x is below M/p if and only if $bp^2 - p(Mc + b) + 2M > 0$. This inequality will be discussed later.

Derivative of demand of x with respect to its price: When it is interior and the bliss point is unaffordable, $x^d(p, M)$ changes with p in the following way

$$\frac{\partial x^d(p, M)}{\partial p} = \frac{(b - 2M)p^2 - 2(b - cM)p + [b(c - 1) + M(2 - c^2)]}{2(1 - cp + p^2)^2} \tag{11}$$

[6] Note that $1 + p^2 - cp > 0$ because $2 > c$.

which is positive if and only if

$$(b-2M)p^2 - 2(b-cM)p + [c(bc-M) - (b-2M)] > 0 \tag{12}$$

Now adopt the following changes of variables:

$$w = b - 2M \tag{13}$$

$$z = b - Mc \tag{14}$$

$$u = \frac{z}{w} \tag{15}$$

Remember that $c < 2$ is needed. Therefore $z > w$. Inequalities (9) through (12) now become

$$p > 1 - \frac{w+z}{b} = p^\star \tag{16}$$

$$w\,(u-p) > 0 \tag{17}$$

$$w\left(p^2 - 2up + cu - 1\right) > 0 \tag{18}$$

Remember that $p^\star > 0$ if and only if $b < M(2+c)$. Focus on the left hand side of inequality (18), which is a polynomial of second degree in p. It intersects the horizontal axis at

$$\tilde{p} = \left(u - \sqrt{u^2 + 1 - cu}\right) \tag{19}$$

$$\hat{p} = \left(u + \sqrt{u^2 + 1 - cu}\right) \tag{20}$$

and therefore it is positive if and only if $w > 0$ and $p < \tilde{p}$ or $p > \hat{p}$, or $w < 0$ and $p \in (\tilde{p}, \hat{p})$ provided that $\tilde{p}$ and $\hat{p}$ are positive. While $\hat{p}$ is always non negative, we have that the sign of $\tilde{p}$ depends on the parameter values. Specifically:

$$\tilde{p} < 0 \Leftrightarrow \begin{cases} b < cM \text{ and } u \in [0, 1/c) \\ b \in [cM, 2M) \\ b > 2M \text{ and } u \in [0, 1/c) \end{cases} \tag{21}$$

Finally observe that $\tilde{p} < u < \hat{p}$. The following cases need to be analyzed:

No Giffen good when $b < cM$. The inequality $b < cM$ implies that: $z < 0$, $w < 0$, $u \in (0, 1)$, and $p^\star > 1 > u$. The conditions for which $x^d(p, M)$ is increasing in p are given by

$$\begin{aligned} p &> p^\star \\ p &> u \\ p &\in (\tilde{p}, \hat{p}) \end{aligned}$$

However $p^\star > \hat{p}$, and so in this case $x^d(p, M)$ cannot have the Giffen property. In fact $p^\star > \hat{p}$ if and only if

$$1 - \frac{w + z}{b} > u + \sqrt{u^2 + 1 - cu} \tag{22}$$

which can be rewritten as

$$(1 - u) - \frac{w + z}{b} > \sqrt{u^2 + 1 - cu} \tag{23}$$

As both sides of the inequality are positive I can take powers. This gives the following condition (after the immediate simplifications)

$$-2u + \left(\frac{w + z}{b}\right)^2 - 2(1 - u)\frac{w + z}{b} > -cu \tag{24}$$

These are the steps needed to simplify the inequality

$$\left(\frac{w+z}{b}\right)\left(\frac{w+z}{b}-2\right) > u\left(2-c-2\frac{w+z}{b}\right)$$
$$-\left(\frac{w+z}{b}\right)M\left(\frac{2+c}{b}\right) > u\left(\frac{2b-cb-2w-2z}{b}\right) \tag{25}$$

The right hand side can be further manipulated

$$u\left(\frac{2b-cb-2w-2z}{b}\right) = u\left(\frac{2b-cb-2w-2b+2cM}{b}\right) =$$
$$u\left(\frac{-c(b-2M)-2w}{b}\right) = -uw\left(\frac{c+2}{b}\right) = -z\left(\frac{2+c}{b}\right) \tag{26}$$

Therefore inequality (25) becomes

$$-\left(\frac{w+z}{b}\right)M\left(\frac{2+c}{b}\right) > -z\left(\frac{2+c}{b}\right)$$
$$-\left(\frac{w+z}{b}\right)M > -z$$
$$(w+z)M < zb$$
$$wM < z(b-M)$$
$$wM - zM < z(b-2M)$$
$$M(w-z) < zw$$
$$-M^2(2-c) < zw \tag{27}$$

which is always true since $wz > 0$ and $2 > c$.

No Giffen good when $cM < b < 2M$. The condition $cM < b < 2M$ implies that: $z > 0$, $w < 0$, $u < 0$, $p^\star > 0$, and $\tilde{p} < 0$. The conditions for which $x^d(p, M)$ is increasing in p boil down to

$$p > p^\star$$
$$p \in (0, \hat{p})$$

However $p^\star > \hat{p}$, and so even in this case $x^d(p, M)$ cannot have the Giffen property. In fact $p^\star > \hat{p}$ can still be written as

$$-M^2(2-c) < zw \tag{28}$$

where now $zw < 0$. By further manipulating the inequality we have

$$M^2(2-c) > (b - Mc)(2M - b) \tag{29}$$

Inequality (29) simplifies to

$$b^2 - b(2+c)M + M^2(2+c) > 0 \tag{30}$$

which is always true since the minimum value of the above polynomial of second degree in b is give by $M^2(2+c)(2-c) > 0$.

Possible Giffen good when $2M < b < (2+c)M$**.** The condition $2M < b < (2+c)M$ implies that: $w > 0$, $z > 0$, $u > 1$, and $p^\star > 0$. The conditions for which $x^d(p, M)$ is increasing in p are given by

$$p > p^\star$$
$$p < u$$
$$p < \tilde{p} \text{ or } p > \hat{p}$$

Since $\hat{p} > u > \tilde{p}$, we have that the only possible configuration of parameters that generates an increasing demand for x is given by

$$p \in (p^\star, \tilde{p}) \tag{31}$$

given that, in addition, $\tilde{p} > p^\star$. In fact $\tilde{p} > p^\star$ if and only if

$$u - \sqrt{u^2 + 1 - cu} > 1 - \frac{w+z}{b} \tag{32}$$

which can be rewritten as

$$(u-1)+\frac{w+z}{b}>\sqrt{u^2+1-cu} \tag{33}$$

As both sides are positive, it is possible to take squares and obtain

$$-2u+\left(\frac{w+z}{b}\right)^2+2(u-1)\frac{w+z}{b}>-cu \tag{34}$$

which is the same as inequality (24), which we know holds if and only if

$$-M^2(2-c)<zw \tag{35}$$

and it is true since $zw>0$ and $2>c$.

Possible Giffen good when $b>(2+c)M$**.** The condition $b>(2+c)M$ implies that: $w>0$, $z>0$, $u>1$, and $p^\star<0$. The conditions for which $x^d(p,M)$ is increasing in p are given by

$$p<u$$
$$p<\tilde{p},\quad p>\hat{p}$$

Since $\hat{p}>u>\tilde{p}$, we have that the only possible configuration of parameters that generates an increasing demand for x is given by

$$p\in(0,\tilde{p}) \tag{36}$$

provided that $\tilde{p}>0$. We know that this happens whenever $u>\frac{1}{c}$, i.e. when

$$b(c-1)+M(2-c^2)>0 \tag{37}$$

Few other possibilities need to be analyzed:

- $c < 1$. Thus (37) becomes

$$b < M\frac{2-c^2}{1-c} \tag{38}$$

As $2 + c < \frac{2-c^2}{1-c}$ we have that $x^d(p, M)$ is increasing whenever

$$M(2+c) < b < M\tfrac{2-c^2}{1-c} \tag{39}$$

and $p \in (0, \tilde{p})$.

- $1 \leq c < \sqrt{2}$. Thus (37) is always true and therefore $x^d(p, M)$ is increasing whenever

$$b > M(2+c) \tag{40}$$

and $p \in (0, \tilde{p})$.

- $\sqrt{2} \leq c < 2$. Thus (37) becomes

$$b > M\frac{c^2-2}{c-1} \tag{41}$$

Since $2 + c > \frac{c^2-2}{c-1}$ we have that $x^d(p, M)$ is increasing whenever

$$b > M(2+c) \tag{42}$$

and $p \in (0, \tilde{p})$.

Wrapping things up, at this stage we have found that the following configurations are possible candidates to generate a Giffen demand:

1. $p \in (0, \tilde{p})$, $1 \leq c < 2$ and $M < \frac{b}{2+c}$;
2. $p \in (0, \tilde{p})$, $c < 1$ and $M \in \left(\frac{b(1-c)}{2-c^2}, \frac{b}{2+c}\right)$;
3. $p \in (p^\star, \tilde{p})$, $M \in \left(\frac{b}{2+c}, \frac{b}{2}\right)$.

Remember also that in these cases we have also that: $z > 0$, $w > 0$, $u > 1$ and $cu > 1$.
Demand of x must be below M/p.

Demand of x is below M/p if and only if

$$bp^2 - p(Mc + b) + 2M > 0 \tag{43}$$

This inequality is always true whenever $M \in (M_1, M_2)$, where

$$M_1 = \frac{b(4-c)}{c^2} - \frac{2b}{c^2}\sqrt{2(2-c)} \tag{44}$$

and

$$M_2 = \frac{b(4-c)}{c^2} + \frac{2b}{c^2}\sqrt{2(2-c)} \tag{45}$$

Moreover, whenever $M \leq M_1$ or $M \geq M_2$, inequality (43) is true if and only if $p < \bar{p}$ or $p > \bar{\bar{p}}$, where

$$\bar{p} = \frac{b + cM - \sqrt{(b+cM)^2 - 8bM}}{2b} \tag{46}$$

and

$$\bar{\bar{p}} = \frac{b + cM + \sqrt{(b+cM)^2 - 8bM}}{2b} \tag{47}$$

Remark that M_2 is decreasing in $c \in (0,2)$ and $\lim_{c \to 2^-} M_2 = b/2$. Thus $M_2 > b/2$ for any $c \in (0,2)$.

In addition, as $M < b/2$ is a necessary condition for x to be Giffen, we can focus only on the threshold value M_1. Now observe that M_1 is, instead, increasing in $c \in (0,2)$ with $\lim_{c \to 0^+} M_1 = 1/8$ and $\lim_{c \to 2^-} M_1 = b/2$. Finally note that $b/(2+c)$ is decreasing in c and ranges from $b/2$ to $b/4$ in $(0,2)$, while $b(1-c)/(2-c^2)$ is decreasing with a vertical asymptote at $c = 1/2$, and assumes value $1/2$ at $c = 0$ and $c = 2$ and value 0 at $c = 1$. We can see that:

$$M_1 < \frac{b}{2} \quad \text{for any} \quad 0 < c < 2 \tag{48}$$

$$M_1 < \frac{b}{2+c} \quad \text{for any} \quad 0 < c < \sqrt{3} \tag{49}$$

$$M_1 < \frac{b(1-c)}{2-c^2} \quad \text{for any} \quad 0 < c < \frac{\sqrt{17}-1}{4} \text{ or } \sqrt{2} < c < 2 \tag{50}$$

Finally, we have that:

1. $c \in (0,2)$ implies $\bar{p} > p^{\star}$;

2. $M \leq M_1$ implies $\bar{\bar{p}} > \tilde{p}$;

3. The relation between $\bar{p}$ and $\tilde{p}$ is uncertain.

Point 1 can be seen by brute force algebra. Point 2 follows by observing that: $\bar{\bar{p}}$ is decreasing in M since $c \in (0,2)$, while $\tilde{p}$ is increasing in M, and $\bar{\bar{p}}$ evaluated at M_1 is larger than $\tilde{p}$ evaluated at M_1. In fact

$$\frac{\partial \bar{\bar{p}}}{\partial M} = \frac{1}{2b}\left(c + \frac{b(c-4)+c^2M}{\sqrt{(b+cM)^2-8cM}}\right) < 0 \tag{51}$$

since

$$\begin{aligned} c &< \frac{b(4-c)-c^2M}{\sqrt{(b+cM)^2-8bM}} \\ c\sqrt{(b+cM)^2-8bM} &< b(4-c)-c^2M \\ b^2c^2+2bc^3M+c^4M^2-8bc^2M &< 16b^2-8b^2c+b^2c^2+c^4M^2-8bc^2M+2bc^3M \\ 0 &< 16b^2-8b^2c \end{aligned} \tag{52}$$

which is true because $0 < c < 2$. Next,

$$\frac{d\tilde{p}}{dM} = \frac{du}{dM}\left[1 - \frac{2u-c}{2\sqrt{u^2-cu+1}}\right] > 0 \tag{53}$$

In fact $1 - \frac{2u-c}{2\sqrt{u^2-cu+1}} > 0$ since $c \in (0,2)$, and

$$\frac{du}{dM} = \frac{b(2-c)}{(b-2M)^2} > 0 \tag{54}$$

Finally observe that $\bar{\bar{p}}$ evaluated at M_1 is given by $\frac{2-\sqrt{4-2c}}{c}$ while $\tilde{p}$ evaluated at M_1 is given by

$$\frac{2\left(2+c+\sqrt{4-2c}\right)-\sqrt{(2-c)\left(2c^2+15c+34+(2c+8)\sqrt{4-2c}\right)}}{6+c} \tag{55}$$

and the former is bigger than the latter if and only if

$$\left(2-\sqrt{4-2c}\right)(6+c) > 2c\left(2+\sqrt{4-2c}+c\right)-c\sqrt{(2-c)\left(2c^2+15c+34+(2c+8)\sqrt{4-2c}\right)} \tag{56}$$

that is

$$c\sqrt{(2-c)\left(2c^2+15c+34+(2c+8)\sqrt{4-2c}\right)} > 2(c^2+c-6)+3(2+c)\sqrt{4-2c} \tag{57}$$

as the right hand side of the inequality is positive I can take squares and obtain (after some simplifications)

$$c^2\left(2c^2+15c+34+2(c+4)\sqrt{4-2c}\right) > 12(c^2+5c+6)\sqrt{4-2c}+4c^3-2c^2-84c-144 \tag{58}$$

which can be manipulated further into

$$2c^4+11c^3+36c^2+84c+144 > 2\sqrt{4-2c}\left(-c^3+2c^2+30c+36\right) \tag{59}$$

Again both sides of the inequality are positive and so I can take squares and obtain

$$\begin{aligned} &4c^8+44c^7+265c^6+1128c^5+3720c^4+9216c^3+17424c^2+24192c+20736 > \\ &\qquad -8c^7+48c^6+384c^5-1280c^4-7584c^3-576c^2+24192c+20736 \end{aligned} \tag{60}$$

which simplifies to

$$c^2(6+c)^2(5+2c)^2(c^2-4c+20) > 0 \tag{61}$$

which is true for any $c \in (0,2)$.

These considerations allow us to conclude that $\bar{\bar{p}} > \tilde{p}$.

The classification in the proposition then follows by adding these new conditions to 1 through 3 on page 20.

Now focus on the sign of $y^d(p,M)$. It can be studied by looking at the budget constraint, whose total derivative with respect to p gives

$$\frac{\partial y^d(p,M)}{\partial p} = -\left(x^d(p,M) + p\frac{\partial x^d(p,M)}{\partial p}\right) \tag{62}$$

Therefore whenever x is a Giffen good, an increase in p decreases the demand of y. However the sign of this derivative is, in general, uncertain. In fact

$$\frac{\partial y^d(p,M)}{\partial p} = \frac{-4Mp - bcp^2 + [b(p^2+2p-1) + cM(p^2+1)]}{2(cp-(p^2+1))^2} \tag{63}$$

The denominator is always positive. So to understand the sign of this derivative we need to look at the numerator. It can be rewritten as

$$p^2(cM + b - bc) + 2p(b-2M) - (b - cM) \tag{64}$$

which, after some manipulation, becomes (remember that $w = b - 2M$, $z = b - cM$, and $u = z/w$)

$$w\left[p^2(u-c) + 2p - u\right] \tag{65}$$

Therefore the zeros of the polynomial of degree two represented by (65) are given by

$$\underline{p} = -\frac{1 + \sqrt{u^2 + 1 - cu}}{u - c} \tag{66}$$

and

$$\underline{\underline{p}} = -\frac{1 - \sqrt{u^2 + 1 - cu}}{u - c} \tag{67}$$

The denominator, which determines also the concavity of the coefficient of the polynomial, is positive whenever $u > c$. The numerator of $\underline{p}$ is always negative. The numerator of $\underline{\underline{p}}$ is positive if and only if $u < 0$ or $u > c$. Therefore

$$\frac{\partial y^d(p,M)}{\partial p} > 0 \Leftrightarrow \begin{cases} u < 0 & p \in (0, \underline{p}) \\ u \in [0, c) & p \in (\underline{\underline{p}}, \underline{p}) \\ u > c & p > \underline{\underline{p}} \end{cases} \tag{68}$$

Now consider the following: $c \in \left(0, \frac{\sqrt{17}-1}{4}\right)$; $M \in \left(\frac{b(4-c)-2b\sqrt{2(2-c)}}{c^2}, \frac{b(1-c)}{2-c^2}\right)$. It follows that $y^d(p, M)$ is interior for any $p < u$, and that $u > c$. [7] Note also that $\underline{\underline{p}} < u$ since $u > c$. Thus from (68) we have that $y^d(p, M)$ has a minimum at $p = \underline{\underline{p}}$, and so $\frac{dy^d(p,M)}{dp}$ changes sign in $p \in (0, u)$. Last remark that for these parameters specifications $x^d(p, M)$ is decreasing in p. $\square$

Proof of Proposition 3. The first part is discussed in the main text, so now focus on the sufficient condition for the upper bound of the share of the expenditure on x. The relevant part is when the bliss point is unaffordable, and therefore all the income is spent purchasing both commodities. In this case the share of income that is spent on commodity x is less than 1/2 if and only if $px^d(p, M) < y^d(p, M)$, i.e. if and only if

$$\begin{aligned}
(b - Mc)p - (b - 2M)p^2 &< (bp - Mc)p - (bp - 2M) \\
bp - bp^2 + 2Mp^2 &< bp^2 - bp + 2M \\
2bp - 2bp^2 + 2Mp^2 - 2M &< 0 \\
2p[M(p + 1) - bp] - 2[M(p + 1) - bp] &< 0 \\
2(p - 1)[M(p + 1) - bp] &< 0 \qquad (69)
\end{aligned}$$

Since we are assuming x to be Giffen, then $p < 1$. Hence inequality (78) holds if and only if

$$M(p + 1) - bp > 0 \qquad (70)$$

This condition can be rewritten as

$$M > p(b - M) \qquad (71)$$

Since x is a Giffen good, then $b > M$ and thus the inequality can be written as

$$p < \frac{M}{b - M} = \breve{p} \qquad (72)$$

[7] Remember that $y^d(p, M)$ is interior if and only if $x^d(p, M)$ is interior.

One can verify that $\breve{p} > p^\star$ and thus the next step is to show under which parameter conditions $\breve{p} > \min\{\tilde{p}, \bar{p}\}$.

The remainder of the proof involves the following steps:

1. Since $\alpha > \beta \Rightarrow \alpha > \min\{\beta, \gamma\}$, discuss first the conditions under which $\breve{p} > \tilde{p}$; this will be true whenever $c \leq 1$, which covers cases 4 to 6 of Proposition 2.

2. Focus on the cases where $c > 1$. Observe that $\bar{p}$ appears only in cases 1, 3 and 8 of Proposition 2 and that in those cases, $\breve{p} < \bar{p}$. Hence the discussion of the condition $\breve{p} > \tilde{p}$ becomes the only relevant one when $c > 1$, and will generate the classification in the proposition.

Let us discuss the inequality $\breve{p} > \tilde{p}$. It is true if and only if

$$\frac{M}{b-M} > \frac{b-Mc}{b-2M} - \sqrt{\left(\frac{b-Mc}{b-2M}\right)^2 - c\frac{b-Mc}{b-2M} + 1} \tag{73}$$

which can be rewritten as

$$\sqrt{\left(\frac{b-Mc}{b-2M}\right)^2 - c\frac{b-Mc}{b-2M} + 1} > \frac{b-Mc}{b-2M} - \frac{M}{b-M} \tag{74}$$

Remark first that the right hand side is positive and therefore it is possible to take squares of both sides. This gives

$$1 + \frac{2M}{b-M}\frac{b-cM}{b-2M} - c\frac{b-cM}{b-2M} - \left(\frac{M}{b-M}\right)^2 > 0 \tag{75}$$

which can be rearranged into

$$\frac{[cM + b(1-c)]\,[(2+c)M^2 - b(2+c)M + b^2]}{(M-b)^2(b-2M)} > 0 \tag{76}$$

The denominator of the left hand side of inequality (76) is always positive when x is a Giffen good. Therefore the sign of inequality (76) is determined by the numerator. Remark that

$$(2+c)M^2 - b(2+c)M + b^2 > 0 \tag{77}$$

Then the sign of (76) depends on the sign of $cM + b(1-c)$, which is always positive whenever $c \leq 1$. Hence expenditure is less than $M/2$ for any $c \leq 1$, which covers cases 4 to 6.

On the other hand, when $c > 1$, inequality (76) is positive whenever

$$M > \frac{b(c-1)}{c} = \bar{M} \tag{78}$$

We have that if $\bar{M}$ is smaller than the lower bound on M needed to have a Giffen demand, then the expenditure on x is less than $M/2$ for any price for which x is a Giffen good. If $\bar{M}$ is included in the values of M needed to have a Giffen demand, then the expenditure on x is less than $M/2$ only for certain values of M. Last, if $\bar{M}$ is larger than the upper bound on M needed to have a Giffen demand, than $\breve{p} < \tilde{p}$ and therefore the expenditure on x is less than $M/2$ only when p is not too big.

Remember that we are considering the instances where $c > 1$. According to Proposition 2, these are covered by cases 1, 2, 3, 7, 8, 9. Of these, 2, 7 and 9 involve just $\tilde{p}$. We consider them first.

In case 2, $\frac{b(c-1)}{c} < \frac{b(4-c)}{c^2} - \frac{2b}{c^2}\sqrt{2(2-c)}$ whenever $1 < c < \sqrt{5} - 1$. When instead $c \in [\sqrt{5}-1, \sqrt{2})$, $\breve{p} > \tilde{p}$ whenever $M \in \left[\frac{b(c-1)}{c}, \frac{b}{2+c}\right)$.

In case 7, the expenditure is always below $M/2$ since $\frac{b(c-1)}{c} < \frac{b(4-c)}{c^2} - \frac{2b}{c^2}\sqrt{2(2-c)}$ whenever $c \in (\sqrt{3}, 2)$.

Finally, in case 9 $\breve{p} > \tilde{p}$ whenever $c \in (1, \sqrt{2})$ or $1 < c \geq \sqrt{2}$ and $M \in \left(\frac{b(c-1)}{c}, \frac{b}{2}\right)$.

Now focus on the remaining cases 1, 3 and 8. We have that $\breve{p} < \bar{p}$ in all of them, and therefore the relevant cases for our analysis are when $\breve{p} \geq \tilde{p}$. We know this is true for any $c \leq 1$, so again one has to consider $c > 1$.

In case 1, we have that $\breve{p} < \tilde{p}$ since $\frac{b(c-1)}{c} > \frac{b}{2+c}$ for any $c > \sqrt{2}$.

Case 3: when $c \in (1, \sqrt{5}-1)$, we have that $\breve{p} > \tilde{p}$ for any $M \in \left(\frac{b(c-1)}{c}, \frac{b(4-c)}{c^2} - \frac{2b}{c^2}\sqrt{2(2-c)}\right)$.

Case 8: We have that $\breve{p} < \tilde{p}$ since $\frac{b(c-1)}{c} > \frac{b(4-c)}{c^2} - \frac{2b}{c^2}\sqrt{2(2-c)}$ whenever $c \in (\sqrt{3}, 2)$.

Wrapping things up we have that the share of income spent on the inferior commodity is bounded from above by 1/2 (cases 4, 5, 6 and 7), it can be bounded if c is not too large (cases 2 and 9) or when it is sufficiently large and income is also sufficiently large (cases 2, 3 and 9) or when prices are not to large (cases 1,and 8). □

Proof of Proposition 4. Suppose x is a Giffen good. The demand of x can be written as

$$x^d(p,M) = \frac{w}{2}\frac{u-p}{p^2+1-cp} \tag{79}$$

where $w = b - 2M > 0$ and $u = \frac{b-cM}{b-2M} > 1$. Its second derivative is thus given by

$$\frac{\partial^2 x^d(p,M)}{\partial p^2} = -\frac{w\left[p^3 - 3p^2u - 3p(1-cu) + c(1-cu) + u\right]}{(p^2+1-cp)^3} \tag{80}$$

which is negative if and only if

$$f(p) = p^3 - 3p^2u - 3p(1-cu) + c(1-cu) + u > 0 \tag{81}$$

Now the logic is as follows: I show that the function is increasing when $p \in (0,\tilde{p})$, and then look at the conditions under which $f(0) > 0$, because this is sufficient (but not necessary) to guarantee concavity of the demand for $p \in (0,\tilde{p})$.
The first derivative of $f(p)$ is

$$f'(p) = 3p^2 - 6pu - 3(1-cu) \tag{82}$$

and it is zero at $p = \tilde{p}$ and $p = \hat{p}$ where

$$\tilde{p} = u - \sqrt{u^2+1-cu} \tag{83}$$

and

$$\hat{p} = u + \sqrt{u^2+1-cu} \tag{84}$$

Remark that $\tilde{p}$ and $\hat{p}$ are the values of p found in the proof of Proposition 2. Thus x is a Giffen good means that $w > 0$ and $p < \min\{\tilde{p},\bar{p}\}$. Finally observe that $f(p)$ is increasing for any $p \in [0,\tilde{p})$ and that, since $\tilde{p} > 0$, we have that $cu > 1$.
Now focus on the value of $f(p)$ at $p = 0$. we have that $f(0) = u(1-c^2) + c > 0$ if and only if on of the following holds:

1. $c \leq 1$;

2. $1 < c < \frac{\sqrt{5}+1}{2}$ and $0 < M < b\frac{c^2-c-1}{c(c^2-3)}$;

3. $c > \sqrt{3}$ and $M > b\frac{c^2-c-1}{c(c^2-3)}$.

Point 1 is sufficient to show that in cases 4, 5 and 6 demand is concave and increasing, since they all involve $c < 1$. Note that $c > \sqrt{3}$ implies that and $b\frac{c^2-c-1}{c(c^2-3)} > \frac{b}{2}$ and therefore point 3 can never hold when x is a Giffen good. Finally by comparing the conditions at point 2 with those on Proposition 2 we can see that the sufficient conditions for concavity do not hold in cases 1, 7 (however see Figure 3 for a counterexample) and 8, whereas they are consistent with

- case 2 for any $c \in (1, \sqrt{2})$ or $c \in \left(\sqrt{2}, \frac{3}{2}\right)$ and $M \in \left(\frac{b(4-c)-2b\sqrt{2(2-c)}}{c^2}, \frac{b(c^2-c-1)}{c^3-3c}\right)$;
- case 3 for any $c \in \left(1, \frac{3}{2}\right)$ or $c \in \left(\frac{3}{2}, \sqrt{3}\right)$ and $M < \frac{b(c^2-c-1)}{c^3-3c}$;
- case 9 for any $c \in (0, \sqrt{2})$ and $M \in \left(\frac{b}{2+c}, \frac{b(c^2-c-1)}{c^3-3c}\right)$.

$\square$

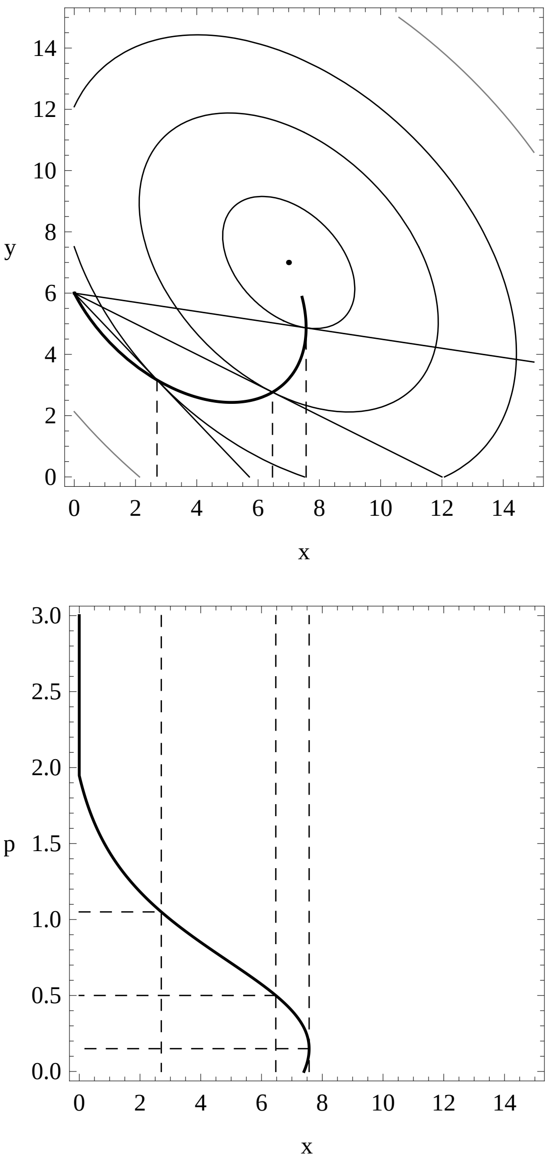

Figure 1: Top: Income consumption path as price of x changes. Bottom: Demand curve of x which shows the Giffen property.

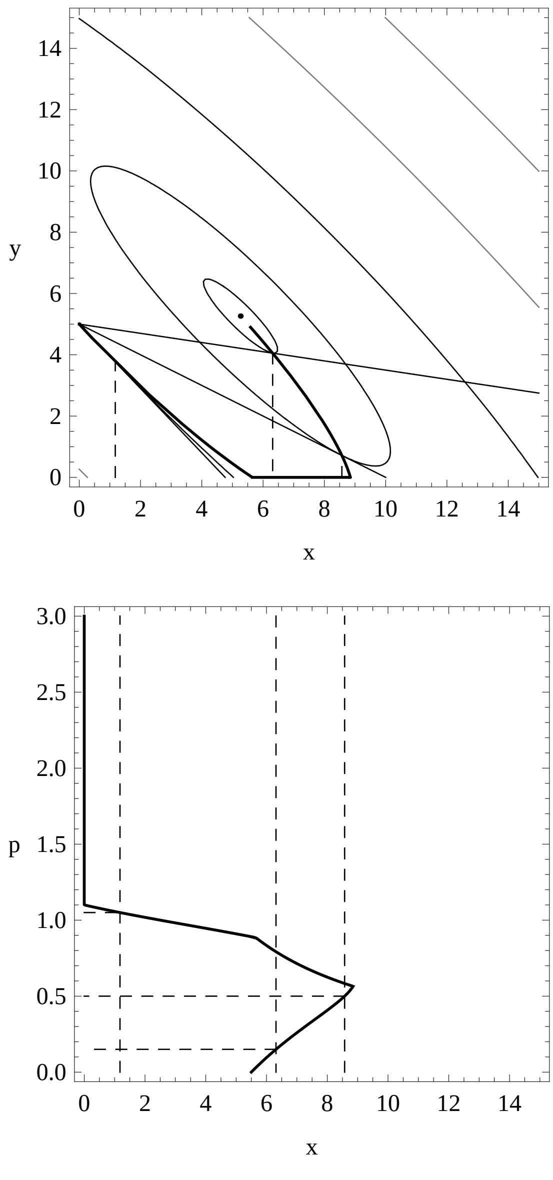

Figure 2: Derivation of demand of x when $b = 20$, $c = 1.8$, $d = 105.263$, and $M = 5$, which corresponds to case 1 in Proposition 2. Top: Income consumption path as price of x changes. Bottom: Demand curve of x.

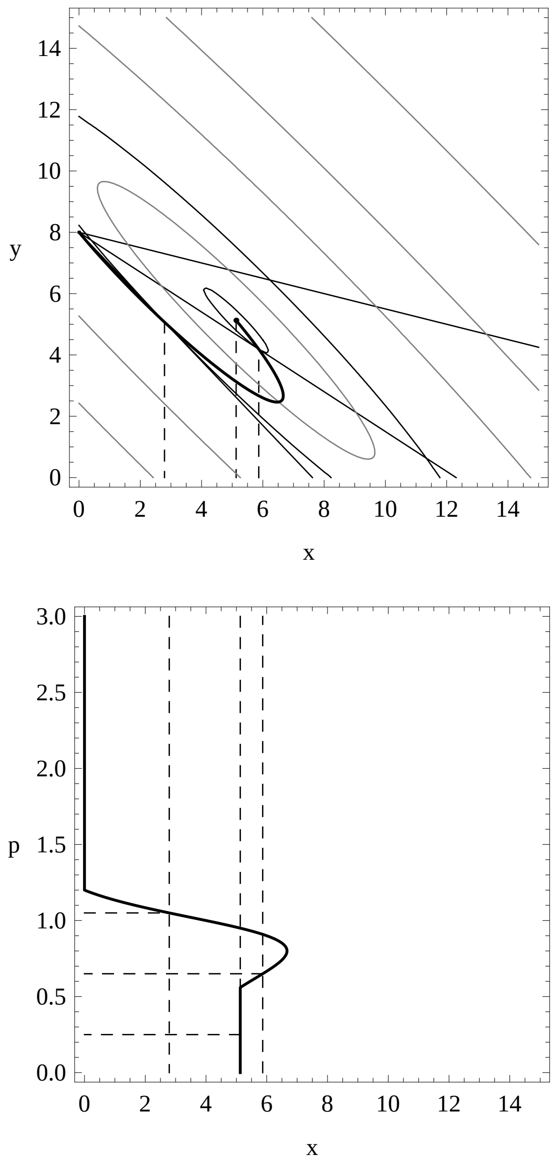

Figure 3: Derivation of demand of x when $b = 20$, $c = 1.9$, $d = 102.564$, and $M = 8$, which corresponds to case 7 in Proposition 2. Top: Income consumption path as price of x changes. Bottom: Demand curve of x.